Group theoretical methods and wavelet theory (coorbit theory and applications)

Hans G. Feichtinger[a]

[a]Faculty of Mathematics, University Vienna, AUSTRIA;

ABSTRACT

Before the invention of orthogonal wavelet systems by Yves Meyer[1] in 1986 Gabor expansions (viewed as discretized inversion of the Short-Time Fourier Transform[2] using the overlap and add OLA) and (what is now perceived as) wavelet expansions have been treated more or less at an equal footing. The famous paper on *painless expansions* by Daubechies, Grossman and Meyer[3] is a good example for this situation. The description of atomic decompositions for functions in modulation spaces[4] (including the classical Sobolev spaces) given by the author[5] was directly modeled according to the corresponding atomic characterizations by Frazier and Jawerth,[6,7] more or less with the idea of replacing the dyadic partitions of unity of the Fourier transform side by uniform partitions of unity (so-called BUPU's, first named as such in the early work on Wiener-type spaces by the author in 1980[8]).

Watching the literature in the subsequent two decades one can observe that the interest in wavelets "took over", because it became possible to construct *orthonormal wavelet systems* with compact support and of any given degree of smoothness,[9] while in contrast the Balian-Low theorem is prohibiting the existence of corresponding Gabor orthonormal bases, even in the multi-dimensional case and for general symplectic lattices.[10] It is an interesting historical fact that* his construction of band-limited orthonormal wavelets (the Meyer wavelet, see[11]) grew out of an attempt to prove the impossibility of the existence of such systems, and the final insight was that *it was not impossible to have such systems*, and in fact quite a variety of orthonormal wavelet system can be constructed as we know by now.

Meanwhile it is established wisdom that wavelet theory and time-frequency analysis are two different ways of decomposing signals in orthogonal resp. non-orthogonal ways. The unifying theory, covering both cases, distilling from these two situations the common group theoretical background lead to the theory of *coorbit spaces*,[12,13] established by the author jointly with K. Gröchenig. Starting from an integrable and irreducible representation of some locally compact group (such as the "ax+b"-group or the Heisenberg group) one can derive families of Banach spaces having natural atomic characterizations, or alternatively a continuous transform associated to it. So at the end function spaces of locally compact groups come into play, and their generic properties help to explain why and how it is possible to obtain (non-orthogonal) decompositions.

While unification of these two groups was one important aspect of the approach given in the late 80th, it was also clear that this approach allows to formulate and exploit the analogy to Banach spaces of analytic functions invariant under the Moebius group have been at the heart in this context. Recent years have seen further new instances and generalizations. Among them shearlets or the Blaschke product should be mentioned here, and the increased interest in the connections between wavelet theory and complex analysis.

The talk will try to summarize a few of the general principles which can be derived from the general theory, but also highlight the difference between the different groups and signal expansions arising from corresponding group representations. There is still a lot more to be done, also from the point of view of applications and the numerical realization of such non-orthogonal expansions.

Keywords: wavelet theory, Gabor analysis, coorbit theory, Besov spaces, frames, Banach frames, Banach Gelfand triples, modulation spaces, atomic decompositions

Further author information: (Send correspondence to Hans G. Feichtinger)
hans.feichtinger@univie.ac.at, Telephone: +431 4277 50696

*As Yves Meyer told me in a short, but highly motivating visit to Paris in February 1987.

Independent Component Analyses, Compressive Sampling, Wavelets, Neural Net, Biosystems, and Nanoengineering XI,
edited by Harold H. Szu, Proc. of SPIE Vol. 8750, 875002 · © 2013 SPIE
CCC code: 0277-786X/13/$18 · doi: 10.1117/12.2018950

1. INTRODUCTION

A good overview of what is nowadays perceived is “Classical Fourier Analysis” is given in the survey talk by Charles Fefferman[14] held at the International Mathematical Congress 1974 which features Calderon-Zygmund operators, $\mathbf{H}^p$-spaces, atomic decompositions and Cotlar's Lemma, and of course Carleson's famous Acta paper,[15] on the *convergence and growth of partial sums of Fourier series*. In fact, in this period the school around E. Stein started to develop systematically ways to describe the smoothness of functions, using either Bessel potentials (allowing to describe fractional smoothness, with Sobolev spaces arising as the natural cornerstones for integer smoothness, positive or negative) or Besov spaces ($\mathbf{B}^s_{p,q}(\mathbb{R}^d)$), which can be viewed as generalized Lipschitz spaces (see Stein's book on *Singular Integrals and Differentiability Properties of functions*[16]). Clearly here the Russian tradition with work of S. Nikolskii (1905 - 2012)[17] and of course even earlier the influential work of S. Sobolev (e.g.[18]). Smoothness is described in a sophisticated way by moduli of continuity (see e.g.[19]), typically based on $\mathbf{L}^p$-norms, because a lot of knowledge had been accumulated in the functional analysis and Fourier analysis community about these spaces, and may not so much because the membership of a function in some $\mathbf{L}^p$-space is such a natural or remarkable property for a given (class of) measurable function.

Another important tool in the background of this development are the remarkable Paley-Littlewood theory, which is at the basis of decompositions of these spaces for all these function spaces. In fact it provides atomic decompositions of tempered distributions with the extra property, that the summability conditions of the corresponding coefficients (which are not uniquely determined) in terms of weighted mixed norm spaces allow to determine the membership of a given function in one of these smoothness spaces (see the work of Frazier and Jawerth[6]). For the pioneers in interpolation theory, Jaak Peetre and Hans Triebel, the *dyadic decompositions on the Fourier transform side* these “Function Spaces” where a crucial tool in identifying the interpolation spaces, using either real or complex interpolation methods, for pairs pairs of Banach spaces over $\mathbb{R}^d$, arising in analysis, which in turn allowed them to verify boundedness results for operators between such function spaces. Clearly the Hausdorff-Young theorem, claiming that the Fourier transform maps $\mathbf{L}^p$ into $\mathbf{L}^q$ (with $1/p + 1/q = 1$) is the prototypical result in this direction.

In this way also the family of so-called Triebel-Lizorkin spaces $\boldsymbol{F}^s_{p,q}(\mathbb{R}^d)$ arose (with potential spaces being special cases, for $q = 2$ within this family). For a systematic summary of all the known properties of these spaces (duality, embedding, traces and much more) the reader my consult the books of Hans Triebel. It was also recognized that the $\mathbf{L}^p$-spaces belong to this family, but only for $1 < p < \infty$, while one should replace $\mathbf{L}^1(\mathbb{R}^d)$ by the Hardy space $\mathbf{H}^1(\mathbb{R}^d)$ for $p = 1$ and its dual, the famous **BMO**-space.

Given the usefulness of these function spaces for many purposes and the multitude of concepts relying of them it is well understandable why wavelet theory, which is well adapted to those function spaces, has gained quickly high recognition in the field of analysis as an important and relatively universal tool. Let us just mention - as typical examples - a few of them:

- theory of PDE in the spirit of L. Hörmander;[20]
- the theory of tempered distributions in the sense of L. Schwartz;[21]
- the theory of pseudo-differential operators[22] with the idea of micro-local analysis in the background;
- the theory of Calderon-Zygmund operators;[23–25]

Nevertheless the reader should be aware, that mathematical analysis provides not only these function spaces, but a huge variety of alternative spaces. I mean this not just for academic considerations, but for the verification of questions of practical interest, also for applied scientists. Despite their obvious relevance in the literature one may promote the idea that perhaps some of those other spaces deserve a more prominent role in our studies. Clearly the $\mathbf{L}^2(\mathbb{R}^d)$ is important, because it is a Hilbert space, and also $\mathbf{L}^1(\mathbb{R}^d)$, because it is the natural domain for the Fourier transform (as long as it is seen as an *integral transform*).

2. FUNCTION SPACES AND OPERATORS

While function spaces are considered often only as an auxiliary tool to describe operators, and the general believe is that one has to choose them from a given relatively small (or large, depending on the knowledge in this field) reservoir, we would like to emphasize here that the choice of the correct setting may be crucial for a good description of operators. Often a certain type of continuous transforms are related atomic decompositions are crucial for the strength of statements that one can make. In this sense the $\mathbf{L}^p$-behavior of the Fourier transform is on the difficult (and content-wise not so satisfactory) side, while on the other hand wavelet expansions make them well suited for the description of Calderon-Zygmund operators[25] while modulation spaces may be useful to describe slowly time-variant channels,[26] as they appear in mobile communication.

2.1 Operators on Families of Normed Spaces

Normally people are happy by either working with individual spaces, say the Hilbert space $\mathcal{H} = \big(\mathbf{L}^2(\mathbb{R}^d), \, \|\cdot\|_2\big)$, or some $\mathbf{L}^p$-space, and to know that certain operators are bounded on such spaces. Think of the Hausdorff Young theorem for the FT, or the boundedness of Fourier multipliers. A typical result of this type is the fact, that for any given $p \in (1, \infty)$ and any rectangular domain $B \subseteq \mathbb{R}^2$ the Fourier multiplier $f \mapsto \boldsymbol{F}^{-1}(\mathbf{1}_B \cdot \boldsymbol{F} f)$. For the $1D$ case this simply means that the ideal low-pass filter, i.e. convolution with the classical SINC kernel is bounded on each individual $\mathbf{L}^p$-space, as long as $1 < p < \infty$, but (unfortunately) with bounds $C_p > 0$ depending on the value of p, and tending to ∞ as $p \to 1$ or $p \to \infty$, with the well known problem of unboundedness on $\big(\mathbf{L}^1(\mathbb{R}^d), \, \|\cdot\|_1\big)$ resp. $\big(\mathbf{L}^\infty(\mathbb{R}^d), \, \|\cdot\|_\infty\big)$.

It is more in the spirit of interpolation theory to view such results not so much as individual triples (a given operator T mapping a Banach space $(\mathbf{B}^1, \, \|\cdot\|^{(1)})$ into another Banach space $(\mathbf{B}^2, \, \|\cdot\|^{(2)})$), but rather as a situation where an operator is defined (maybe in different format, depending on the input) on a whole family of Banach spaces, small and large, and where the user may be interested to find out what can be said about the output. Just think of BIBO-systems, where the guarantee is given that bounded input guarantees bounded output, but then analysis shows that it is also bounded on $\mathbf{L}^1$ (because it must be a convolution operator with a bounded measure μ as kernel), and finally on the whole family of $\mathbf{L}^p$-spaces, now with $1 \leq p \leq \infty$, with a joint upper bound, namely the total variation norm $\|\mu\|_{\mathbf{M}}$.

2.2 Wiener Amalgam Spaces

Among the many possible function spaces of interest let us feature specifically the so-called Wiener amalgam spaces. They carry this name because they go back to Wiener's work on Tauberian theorems. Since he wanted to include also certain unbounded measures in his consideration he made use of a construction quite similar to that of an a finite upper Riemann sum (over the whole real line). Such a construction is quite important as it allows to separate the local from the global effects (e.g. by taking ordinary $\mathbf{L}^p$-norms). A good survey of the state of the art is provided by the article by J. Fournier and J. Stewart[27] in the Bull. Amer. Math. Soc.. They consider only local $\mathbf{L}^p$-norms and global ℓ^q-summability, but list a long number of interesting applications. Among others the work of Busby and Smith[28] on product convolution operators is very instructive. The idea to allow also for similar spaces, but e.g. with the Fourier algebra $\boldsymbol{F}(\mathbf{L}^1(\mathbb{R}^d))$ as a "local component" was the motivation of the introduction of the concept of (at that time names) Wiener-type spaces by the author,[8] which where then used in order to create modulation spaces (introduced by the author in 1983,[4,5,29–31] and subsequently developed in great detail). Nowadays modulation spaces are a well established tool, specifically in time-frequency analysis, and many questions have their natural description in terms of various modulation spaces.

Since the generalization of many concepts (such as continuous linear operators, robustness of constructions against certain perturbations, such as jitter error analysis in sampling theory) are beyond the scope of this short note let us try to communicate the very idea using a comparison.

3. BANACH GELFAND TRIPLES

For a motivation, telling us also a little bit how to make use of Banach Gelfand triples, let us recall some basic facts concerning ordinary numbers resp. the fields to which they belong, and how they are related, practically and also somehow psychologically.

3.1 From rational to real and complex numbers

While mathematicians tend to start from an axiomatic setting, and define the axioms of a field, in a minimalistic way, and derive all the relevant properties from first principles it is more important for engineers to know in which situation which computational method can be applied without producing wrong results. Minimal complexity of the description is preferred over maximal range of validity at the cost of complicated notations. On the other hand over-simplified notations may lead to wrong conclusions and careless argumentation (or discipline) may leave great uncertainties on the reader's side.

So instead of discussing at length what kind of ordered and perhaps non-Archemedian fields there may exist engineering students learn how to use the fields $\mathbf{Q}$ of rationals, the real number system $\mathbb{R}$ and finally the complex numbers $\mathbf{C}$, because Euler's formula allows to make the connection between exponential law and addition theorems for trigonometric functions. Clearly division can be performed exactly within $\mathbf{Q}$ while already in $\mathbb{R}$ one has to do some kind of approximation, while for the complex numbers the inversion is reduced to real number case (e.g. through polar coordinates). Nevertheless it is clear that they all are just "numbers" and that one may identify $2/5 \in \mathbf{Q}$ with $0.4 \in \mathbb{R}$ or $0.4 + 0 \cdot i \in \mathbf{C}$, as needed.

We have explained this example because it can provide a good guidance to the understanding how Banach Gelfand triples work and why they are useful. They also consist of a small reservoir of nice object, the so-called test functions, a bigger one, namely $\left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, which is not only complete but also a Hilbert space. Finally we go beyond this setting, and the introduction of generalized functions comes into play. They are "as real as a complex number", meaning that they are well defined objects, with (hopefully) clear computation rules, which extend the computations done in the smaller domains usually in a unique fashion. Hence operations such as duality pairing through "integrals", namely

$$(f, g) \mapsto \int_{\mathbb{R}^d} f(x) g(x) dx,$$

or convolution, pointwise multiplication or the application of operators (such as the Fourier transform!) are usually quite well defined on the smaller domain, perhaps even using the good old Riemannian integral, and have to be properly extended to the other setting (and hence to all the spaces "in between").

3.2 The Segal Algebra $\mathbf{S}_0(\mathbb{R}^d)$

The first among all of the modulation spaces which had been introduced (already in 1979, and published in 1981[32]) by the author was the Segal algebra $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$, which was in fact introduced as a variant of Wiener's algebra, which is in modern terminology the Wiener amalgam space $\mathbf{W}(C_0, \ell^1)(\mathbb{R}^d)$. This Segal algebra was introduced as the Banach space space $\mathbf{S}_0(\mathbb{R}^d) = \mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1)(\mathbb{R}^d)$, which is sometimes also called Feichtinger's algebra (see the revised version of Reiter's book on Harmonic Analysis, coauthored with I. Stegeman[33]). Hence one can say, $f \in \mathbf{S}_0(\mathbb{R})$ if and only if, after decomposing the function into sum of local pieces (e.g. by using a sequence of triangular functions of constant shape, adding up to the constant 1), the total sum of the absolute values of all the Fourier coefficients (over all positions and all frequencies) is finite. It also should be noted that the same space has been introduced independently at about the same time by J.P. Bertrandias,[34] also as a generalization of standard amalgam spaces as described in the survey by Fournier and Stewart.[27]

This space is a Banach space, containing the Schwartz space of rapidly decreasing functions $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ as a dense subspace, but is contained in all the $\mathbf{L}^p$-spaces. It is also isometrically translation invariant, and this is the reason why it is a *Segal algebra*, following Hans Reiter's definitions, see[35] or the updated version.[33] But it is also Fourier invariant, hence also even isometrically invariant under time-frequency shifts. V. Losert[36] has shown that this algebra (well defined for any LCA group) is uniquely determined by its many good invariance properties. All the classical summability kernels belong to this space, and this expresses clearly that/why this space can be considered as a good space of test functions for harmonic analysis in an abstract setting, but also for signal processing applications. Among others one can show easily that Poisson's formula holds for any $f \in \mathbf{S}_0(\mathbb{R}^d)$ and therefore it is a good vehicle to prove e.g. the Shannon Sampling Theorem resp. the fact that sampling on the time sided corresponds to the periodization on the Fourier side.

For those familiar with Schwartz's theory of tempered distributions one may say, that $\mathbf{S}_0(\mathbb{R}^d)$ is a very good and technically much less challenging space of test functions compared to the Schwartz space, mostly because it is a Banach space. In addition it is absolutely sufficient for most engineering applications (except for PDE applications, and this is just the domain for which this theory was made for[21]).

In fact, the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ which is a so-called nuclear Frechet space, i.e. only a topological vector space with some complicated (although interesting) concept of convergence. This is then needed to define the continuous linear functions, i.e. the linear mappings σ from the test-functions to complex field, which are continuous, or in other words respect convergence. They have to satisfy the following implication: If $f_n \to f_0$ (in $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$!) then one has to be ensured that $\sigma(f_n) \to \sigma(f_0)$ in $\mathbf{C}$.

In contrast the dual space $\mathbf{S}_0{}'(\mathbb{R}^d)$, i.e. the family of continuous linear functionals in $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ are easily defined to be just the linear mappings σ from $\mathbf{S}_0(\mathbb{R}^d)$ to $\mathbf{C}$ with the property that $|\sigma(f)| \leq C\|f\|_{\mathbf{S}_0}$ for all $f \in \mathbf{S}_0(\mathbb{R}^d)$. They can be characterized as the space of tempered distributions which have (e.g. with respect to the Gaussian window) a uniformly bounded short-time Fourier transform. Moreover norm convergence is exactly uniform convergence of the corresponding spectrograms, while the equally important so-called w^*-convergence is nothing else but the less restrictive *uniform convergence over compact sets*, which is good enough for applications. Just think of the fact that even a good audio-recording is representing the piece of music only for the (finite) duration of the piece and in a finite frequency, only up to 20 kHz.

3.3 The Banach Gelfand Triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0{}')(\mathbb{R}^d)$

Although the Banach Gelfand triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0{}')(\mathbb{R}^d)$ was first used systematically in the context of Gabor analysis[37,38] it turned out to be useful in many other situations, in particular in the context of classical Fourier analysis. Not only is $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ a convenient Banach space of bounded, continuous and absolutely Riemann integrable functions, it is also a domain for Poisson's formula $\sum_{k\in\mathbf{Z}} f(k) = \sum_{n\in\mathbf{Z}} \hat{f}(k)$, with absolute convergence on both sides. Plancherel's formula, telling us that $\|f\|_2 = \|\hat{f}\|_2$ can be derived from this and provides us with the fact that the Fourier transform is a unitary automorphism of $\left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$. However, out the outer layer, i.e. in the dual space one can make the claim that the Fourier transform maps pure frequencies (also called *characters* of the LCA group in abstract harmonic analysis) into Dirac (point) measures, which is exactly the continuous analogue of the claim that the FFT is just a change from one basis (unit vectors) to another orthonormal basis (namely the pure frequencies). Unfortunately those building blocks are not in $\mathbf{L}^2(\mathbb{R}^d)$ anymore, and the convergence is not the usual one, but rather the so-called w^*-convergence, which however is well known (at least in principle) from probability theory (where the concept of vague convergence is used, e.g. in the formulation of the central limit theorem).

The concept of Banach Gelfand triple, especially the one based on the Segal algebra $\mathbf{S}_0(\mathbb{R}^d)$ appears to be quite similar to that of the usual Gelfand triple, based on the embedding of the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ into $\mathbf{L}^2(\mathbb{R}^d)$ which in turn sits inside the tempered distributions, but in this setting of a *rigged Hilbert space* the topologies involved are much less transparent. Other cases where a similar setting appears to be useful is in the theory of elliptic partial differential operators, which typically map a Sobolev space such as $\mathbf{H}^1(\mathbb{R}^d)$ into its dual $\mathbf{H}^{-1}(\mathbb{R}^d)$, its dual space. Together they also form a triple (now of Hilbert spaces), but certain constructions (such as the kernel theorem) do *not work* in this setting.

The list of possible applications of the theory of Banach Gelfand triples over $\mathbf{S}_0(\mathbb{R}^d)$ is long. According to my view it grew out of the investigations of coorbit spaces in general and the Heisenberg group setting (with modulation spaces viewed as coorbit space for the Schrödinger representation of the Heisenberg group) and therefore it is not surprising that the main applications are in this area, but some aspects are relevant for a much wider range of problems in analysis (up to the claim that the spaces in this triple might be more useful for applications than $\mathbf{L}^p$-theory, at least in some cases), and in particular for the mathematical foundations of signal processing, in particular for Gabor analysis. For a summary of such results the reader can be referred to the survey paper by E. Cordero, F. Luef and the author.[39]

Let us shortly indicate the most striking results which can be formulated within the context of the Banach Gelfand triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0{}')(\mathbb{R}^d)$.

1. The Fourier transform can be viewed as a *unitary* Gelfand triple isomorphism of the BGT $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0{}')(\mathbb{R}^d)$, which means that it is unitary at the Hilbert space level (Plancherel's theorem), but also leaves the other

layers invariant, with the additional property of mapping the dual space $\mathbf{S}_0{}'(\mathbb{R}^d)$ into itself, *both* in the norm and the w^*-topology, i.e. preserving w^*- convergence (for bounded sequences in $(\mathbf{S}_0{}'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0{}'})$);

2. There is a kernel theorem, which extends the classical identification of Hilbert-Schmidt operators with $\mathbf{L}^2(\mathbb{R}^{2d})$-integral kernels. The operators with kernels in $\mathbf{S}_0(\mathbb{R}^{2d})$ are exactly the regularizing operators, i.e. the bounded operators mapping bounded, w^*-convergent sequences in $\mathbf{S}_0{}'(\mathbb{R}^d)$ into norm convergent sequences in $\big(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\big)$. In contrast, general bounded linear operators (i.e. linear systems or channels) from $\mathbf{S}_0(\mathbb{R}^d)$ into $\mathbf{S}_0{}'(\mathbb{R}^d)$ have still a distributional kernel in $\mathbf{S}_0{}'(\mathbb{R}^{2d})$; these are not just exotic operators, but ordinary multiplication operators or convolution operators are typically only in this larger class[†].

4. COORBIT SPACES AND ATOMIC DECOMPOSITION

As already mention coorbit space theory[13] arose as an attempt to isolate the common aspects of e.g. the continuous short-time Fourier transform and the continuous wavelet transform, resp. the expansions of functions or distributions using either Gabor system or wavelet bases or frames. The common aspect of both of these settings (as well as of many similar situations) is the group action in the background, which allows to produce a setting comparable to coherent states.[40–42] One just needs a so-called integrable, irreducible group representation of some locally compact group in order to get started, and as we see more and more there are plenty of such groups, aside of those mentioned above, e.g. recently for example the Blaschke group.[43–45]

5. MODULATION SPACES AND THE HEISENBERG GROUP

The theory of modulation spaces, with the STFT (Short-time Fourier transform) as a continuous transform is connected to the so-called Schrödinger representation of the Heisenberg group,[46] which in fact is a so-called *projective representation* only, i.e. one has to account for suitable phase factors when one computes the composition of two TF-shift operators. From the engineering point of view the atoms of the corresponding (Gabor) expansion of signals are simply discrete subfamilies indexed by $(\lambda_i)_{i\in I}$ within the continuous family of coherent states, obtained by moving a general non-zero atom (such as the Gauss-function) in phase space, by applying a for each such $\lambda = (t, \omega)$ the TF-shift $\pi(\lambda) = M_\omega T_t$. In Gabor analysis the typical choice of such a family would be a lattice of the form $a\mathbf{Z} \times b\mathbf{Z}$. More recently more general (non-separable resp. non-symplectic) lattices came into discussion, and in each case one looks for an efficient computation of the canonical dual window $\tilde{g}$ window. Again group theory is a strong argument allowing meanwhile to find efficient algorithms. First of all the frame operator

$$S \mapsto \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g$$

commutes with all the TF-shifts used, i.e. with $\pi(\lambda), \lambda \in \Lambda$, and consequently has the so-called Janssen representation (from which the Wexler-Raz principle is derived). Even in fairly complicated situations where direct computations would be a bit complicated the coefficients for the contributions different from the identity operator in this representation allows to guarantee the invertibility of the Gabor frame operator $S = S(g, \Lambda)$, in fact not only as an operator on the Hilbert space $\mathbf{L}^2(\mathbb{R}^d)$, but also simultaneously on each of the layers of the Banach Gelfand triple, in particular at the $\mathbf{S}_0$-level. Corresponding numerical considerations can be found in recent papers.[47,48]

6. SHEARLETS AND THE SHEARLET GROUP

Among the more recent examples of coorbit theory, which came from applications and was motivated by the need to have needle-like building blocks for image processing applications (similar to curvelets,[49,50] or ridgelets[51]) the invention of shearlets in the last decade gave rise to a new family of frames (see the early papers by G. Kutyniok[52] and D. Labate[53–55]) and several other papers showing the usefulness of shearlets for image processing applications, or for the description of Fourier integral operators.

Once it had been clarified that there is a shearlet group in the background of the continuous shearlet transform the existence of coorbit theory was of course highly motivating for a group of researchers around Stephan Dahlke, to establish the corresponding theory of shearlet spaces, atomic decompositions and so on in full details.[56–59]

[†]No invertible operator can be Hilbert Schmidt, because a compact operator never has a bounded inverse.

7. LITERATURE, REFERENCES

In this short section we just recollect a few influential papers (at least to the author) of this time. Although the manuscript[29] was around for quite a while, but still unpublished[30] it was clear that there are some common features between atomic decompositions of modulation spaces[5] in the spirit of Gabor (although the term and connection to the work of Gabor was not clear at that time and the decomposition of function using building blocks *of constant shape*.[60] Within the theory of function spaces (in the spirit of Triebel and Peetre) the work on atomic decompositions by Frazier and Jawerth[6,61,62] (based on the Littlewood-Paley theory) was most influential. An indication of the usefulness of this transform was provided in.[63] The paper on *painless decompositions*[64] was emphasizing the analogy between wavelet theory and time-frequency methods.

Of course this is also a place to emphasize the role of the pioneers in *interpolation theory*, namely Jaak Peetre[65] and Hans Triebel with their books.[66–69] In fact, it was Jaak Peetre's paper[70] which was providing the name *coorbit spaces* in contraposition to orbit spaces, which reminds more of atomic decompositions, with building blocks (like coherent states) being obtained as the *orbit* of a given atom (mother wavelet, Gabor atom, etc.) under a certain unitary (projective) group representation.

Furthermore, some of this papers (providing early version of *atomic decompositions*, e.g. in the context of Moebius invariant Banach spaces of analytic functions[71,72]) had a great influence on the creation of coorbit space theory, as well as the Asterisque work of R. Coifman and R. Rochberg.[73]

ACKNOWLEDGMENTS

The author would like to thank my predecessor, Prof. John J. Benedetto, for suggesting my name as a wavelet pioneer. In fact, we got into contact at the time when two of his PostDocs, Chris Heil and David Walnut had prepared their now well cited survey article.[74] It was a summary of their PhD theses[75,76] and had great impact in the community. Dave Walnut's book[77] is also mentioned here. Another valuable book going back to this time, although refined over time, is the Basis Primer by C. Heil.[78]

I had the pleasure of spending the academic year 1989/1990 at College Park/MD, and met in that same year Ingrid Daubechies at ATT (with the paper on Wilson bases as a result[31] as a result), as well as Dave Donoho in Berkeley (who just started to be interested in wavelets, I knew his work on uncertainty[79]) and Gilbert Strang at MIT, giving seminar talks at those places (as well as at the MITRE corporation), and starting the work on irregular sampling, using the analogy between band-limited functions and continuous wavelet transforms (resp. STFTs). It was also at that time that the connections between sampling theory and frame theory become more clear.[80] At that time the concept of Banach frames have been developed[81] (nowadays I would favour Banach frames for Banach Gelfand triples, emphasizing the role of solidity at the level of sequence spaces, which in turn typically implies unconditional convergence of the partial sums of the coorbit decompositions).

REFERENCES

[1] Meyer, Y., "De la recherche petrolière à la géometrie des espaces de Banach en passant par les paraproduite. (From soil investigation to geometry of Banach spaces via paraproducts).," (1986).

[2] Allen, J. B. and Rabiner, L. R., "A unified approach to short-time Fourier analysis and synthesis," *Proc. IEEE* **65,**(11), 1558–1564 (1977).

[3] Daubechies, I. and Grossmann, A., "Frames in the Bargmann Hilbert space of entire functions.," *Commun. Pure Appl. Anal.* **41**(2), 151–164 (1988).

[4] Feichtinger, H. G., "Modulation Spaces: Looking Back and Ahead," *Sampl. Theory Signal Image Process.* **5**(2), 109–140 (2006).

[5] Feichtinger, H. G., "Atomic characterizations of modulation spaces through Gabor-type representations," in [*Proc. Conf. Constructive Function Theory*], *Rocky Mountain J. Math.* **19**(1), 113–126 (1989).

[6] Frazier, M. and Jawerth, B., "Decomposition of Besov spaces," *Indiana Univ. Math. J.* **34**, 777–799 (1985).

[7] Frazier, M. and Jawerth, B., "The φ-transform and applications to distribution spaces.," in [*Function Spaces and Applications, Proc US-Swed Semin, Lund/Swed, Lect Notes Math 1302, 223-246*], (1988).

[8] Feichtinger, H. G., "Banach convolution algebras of Wiener type," in [*Proc. Conf. on Functions, Series, Operators, Budapest 1980*], *Colloq. Math. Soc. Janos Bolyai* **35**, 509–524, North-Holland, Amsterdam, Eds. B. Sz.-Nagy and J. Szabados. ed. (1983).

[9] Daubechies, I., "Orthonormal bases of compactly supported wavelets.," *Commun. Pure Appl. Anal.* **41**(7), 909–996 (1988).

[10] Gröchenig, K., Han, D., Heil, C., and Kutyniok, G., "The Balian-Low theorem for symplectic lattices in higher dimensions," *Appl. Comput. Harmon. Anal.* **13**(2), 169–176 (2002).

[11] Lemarié, P. G. and Meyer, Y., "Ondelettes et bases hilbertiennes. (Wavelets and Hilbert bases).," *Rev. Mat. Iberoam.* **2**, 1–18 (1986).

[12] Feichtinger, H. G. and Gröchenig, K., "A unified approach to atomic decompositions via integrable group representations," *Lect. Notes in Math.* **1302**, 52–73 (1988).

[13] Feichtinger, H. G. and Gröchenig, K., "Banach spaces related to integrable group representations and their atomic decompositions, I," *J. Funct. Anal.* **86**(2), 307–340 (1989).

[14] Fefferman, C., "Recent progress in classical Fourier analysis.." Proc. int. Congr. Math., Vancouver 1974, Vol. 1, 95-118 (1975). (1975).

[15] Carleson, L., "On convergence and growth of partial sums of Fourier series," *Acta Math.* **116**, 135–157 (1966).

[16] Stein, E. M., [*Singular Integrals and Differentiability Properties of Functions*], Princeton University Press, Princeton, N.J. (1970).

[17] Nikol'skij, S. M., [*Approximation of Functions of Several Variables and Imbedding Theorems. Translated from the Russian by J. M. Danskin.*], Die Grundlehren der mathematischen Wissenschaften. Band 205. Berlin-Heidelberg-New York: Springer-Verlag. VIII, 420 p. DM 108.00 (1975).

[18] Sobolev, S., [*Partial Differential Equations of Mathematical Physics*], Oxford-London-New York-Paris-Frankfurt: Pergamon Press. X, 430 p. (1964).

[19] Ditzian, Z. and Totik, V., [*Moduli of Smoothness*], Springer Series in Computational Mathematics, 9. New York etc.: Springer- Verlag. IX (1987).

[20] Hörmander, L., [*The Analysis of Linear Partial Differential Operators II: Differential Operators with Constant Coefficients.*], Springer, Berlin, Heidelberg, New York (1983).

[21] Schwartz, L., [*Théorie des Distributions. (Distribution Theory). Nouveau Tirage. Vols. 1.*], Paris: Hermann. xii, 420 p. FF 230.00 (1957).

[22] Hörmander, L., [*The analysis of linear partial differential operators. III: Pseudo-differential operators*], Grundlehren der Mathematischen Wissenschaften, 274, Springer (1985).

[23] Journe, J.-L., [*Calderon-Zygmund Operators, Pseudo-differential Operators and the Cauchy Integral of Calderon.*], Springer, Berlin, Heidelberg, New York, Tokio (1983).

[24] Meyer, Y., [*Ondelettes et Operateurs II: Operateurs de Calderon-Zygmund. (Wavelets and Operators II: Calderon-Zygmund Operators).*], Hermann, Editeurs des Sciences et des Arts, Paris (1990).

[25] Meyer, Y. and Coifman, R. R., [*Wavelets: Calderon–Zygmund and Multilinear Operators*], no. 48 in Cambridge studies in advanced mathematics, Cambridge University Press, Cambridge (1997).

[26] Gröchenig, K. and Strohmer, T., "Pseudodifferential operators on locally compact abelian groups and Sjöstrand's symbol class," *J. Reine Angew. Math.* **613**, 121–146 (2007).

[27] Fournier, J. J. F. and Stewart, J., "Amalgams of L^p and ℓ^q," *Bull. Amer. Math. Soc., New Ser.* **13**, 1–21 (1985).

[28] Busby, R. C. and Smith, H. A., "Product-convolution operators and mixed-norm spaces," *Trans. Amer. Math. Soc.* **263**, 309–341 (1981).

[29] Feichtinger, H. G., "Modulation spaces on locally compact Abelian groups," tech. rep. (January 1983).

[30] Feichtinger, H. G., "Modulation spaces of locally compact Abelian groups," in [*Proc. Internat. Conf. on Wavelets and Applications*], Radha, R., Krishna, M., and Thangavelu, S., eds., 1–56, New Delhi Allied Publishers, Chennai, January 2002 (2003).

[31] Feichtinger, H. G., Gröchenig, K., and Walnut, D. F., "Wilson bases and modulation spaces," *Math. Nachr.* **155**, 7–17 (1992).

[32] Feichtinger, H. G., "On a new Segal algebra," *Monatsh. Math.* **92**, 269–289 (1981).

[33] Reiter, H. and Stegeman, J. D., [*Classical Harmonic Analysis and Locally Compact Groups. 2nd ed.*], Clarendon Press, Oxford (2000).

[34] Bertrandias, J.-P., "Espaces $l^p(a)$ et $l^p(q)$," *Groupe de travail d'analyse harmonique* **I**, 1–13, Université scientifique et medicale de Grenoble, laboratoire de mathématique pures associé au c.n.r.s. (1982).

[35] Reiter, H., [*Classical Harmonic Analysis and Locally Compact Groups*], Clarendon Press, Oxford (1968).

[36] Losert, V., "Segal algebras with functional properties," *Monatsh. Math.* **96**, 209–231 (1983).

[37] Feichtinger, H. G. and Zimmermann, G., "A Banach space of test functions for Gabor analysis," in [*Gabor Analysis and Algorithms: Theory and Applications*], Feichtinger, H. G. and Strohmer, T., eds., *Applied and Numerical Harmonic Analysis*, 123–170, Birkhäuser Boston, Boston, MA (1998).

[38] Feichtinger, H. G. and Kozek, W., "Quantization of TF lattice-invariant operators on elementary LCA groups," in [*Gabor analysis and algorithms*], Feichtinger, H. G. and Strohmer, T., eds., *Appl. Numer. Harmon. Anal.*, 233–266, Birkhäuser Boston, Boston, MA (1998).

[39] Cobos, F., Fernandez Cabrera, L., Kühn, T., and Ullrich, T., "On an extreme class of real interpolation spaces.," *J. Funct. Anal.* **256**(7), 2321–2366 (2009).

[40] Bargmann, V., Butera, P., Girardello, L., and Klauder, J. R., "On the completeness of coherent states," *Rep. Math. Phys.* **2**, 221–228 (1971).

[41] Klauder, J. R. and Skagerstam, B.-S., eds., [*Coherent States. Applications in Physics and Mathematical Physics.*], World Scientific, Singapore (1985).

[42] Ali, S., Antoine, J.-P., and Gazeau, J.-P., [*Coherent States, Wavelets and their Generalizations*], Graduate Texts in Contemporary Physics, Springer, New York (2000).

[43] Pap, M., "The voice transform generated by a representation of the Blaschke group on the weighted Berman spaces," *Annales Univ. Sci. Budapest., Sect. Comp.* **33**, 321–342 (2010).

[44] Pap, M., "Properties of the voice transform of the Blaschke group and connections with atomic decomposition results in the weighted Bergman spaces.," *J. Math. Anal. Appl.* **389**(1), 340–350 (2012).

[45] Feichtinger, H. G. and Pap, M., "Hyperbolic Wavelets and Multiresolution in the Hardy Space of the Upper Half Plane," *Blaschke Products and Their Applications: Fields Institute Communications* **65**, 193–208 (2013).

[46] Feichtinger, H. G. and Gröchenig, K., "Gabor wavelets and the Heisenberg group: Gabor expansions and short time Fourier transform from the group theoretical point of view," in [*Wavelets :a tutorial in theory and applications*], Chui, C. K., ed., *Wavelet Anal. Appl.* **2**, 359–397, Academic Press, Boston (1992).

[47] Feichtinger, H. G., Grybos, A., and Onchis, D., "Fast approximation of dual Gabor atoms by masking the adjoint lattice," *preprint* .

[48] de Gosson, M. A. and Onchis, D., "Multivariate symplectic Gabor frames with Gaussian windows," *preprint* (2012).

[49] Candès, E. J. and Demanet, L., "Curvelets and Fourier integral operators.," *C. R. Math. Acad. Sci. Paris* **336**(5), 395–398 (2003).

[50] Candès, E. J. and Donoho, D. L., "New tight frames of curvelets and optimal representations of objects with piecewise c^2 singular-ities.," *Commun. Pure Appl. Anal.* **57**(2), 219–266 (2004).

[51] Candès, E. J. and Donoho, D. L., "Ridgelets: a key to higher-dimensional intermittency?," *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **357**(1760), 2495–2509 (1999).

[52] Kutyniok, G. and Sauer, T., "Adaptive directional subdivision schemes and shearlet multiresolution analysis.," *SIAM J. Math. Anal.* **41**(4), 1436–1471 (2009).

[53] Guo, K. and Labate, D., "Optimally sparse multidimensional representation using shearlets," *SIAM J. Math. Anal.* **39**(1), 298–318 (2007).

[54] Guo, K. and Labate, D., "Representation of Fourier integral operators using shearlets," *J. Fourier Anal. Appl.* **14**(3), 327–371 (2008).

[55] Guo, K. and Labate, D., "Characterization and analysis of edges using the continuous shearlet transform.," *SIAM J. Imaging Sci.* **2**(3), 959–986 (2009).

[56] Dahlke, S., Kutyniok, G., Steidl, G., and Teschke, G., "Shearlet coorbit spaces and associated Banach frames.," *Appl. Comput. Harmon. Anal.* **27**(2), 195–214 (2009).

[57] Dahlke, S., Steidl, G., and Teschke, G., "The continuous shearlet transform in arbitrary space dimensions," *J. Fourier Anal. Appl.* **16**(3), 340–364, (2010).

[58] Dahlke, S., Steidl, G., and Teschke, G., "Shearlet coorbit spaces: Compactly supported analyzing shearlets, traces and embeddings.," *J. Fourier Anal. Appl.* **17**(6), 1232–1255 (2011).

[59] Dahlke, S., Steidl, G., and Teschke, G., "Multivariate shearlet transform, shearlet coorbit spaces and their structural properties.." Kutyniok, Gitta (ed.) et al., Shearlets. Multiscale analysis for multivariate data. Boston, MA: Birkhäuser. Applied and Numerical Harmonic Analysis, 105-144 (2012). (2012).

[60] Grossmann, A. and Morlet, J., "Decomposition of functions into wavelets of constant shape, and related transforms," in [*Mathematics and Physics, Lect Recent Results, Bielefeld/FRG 1983/84*], **1**, 135–165 (1985).

[61] Frazier, M. and Jawerth, B., "A discrete transform and decompositions of distribution spaces.," *J. Funct. Anal.* **93**(1), 34–170 (1990).

[62] Frazier, M. and Jawerth, B., "Applications of the φ and wavelet transforms to the theory of function spaces.," in [*Ruskai, Mary Beth (ed) et al, Wavelets and their Applications Boston, MA Etc: Jones and Bartlett Publishers 377-417*], (1992).

[63] Kumar, A., Fuhrmann, D. R., Frazier, M., and Jawerth, B. D., "A new transform for time-frequency analysis.," *IEEE Trans. Signal Process.* **40**(7), 1697–1707 (1992).

[64] Daubechies, I., Grossmann, A., and Meyer, Y., "Painless nonorthogonal expansions," *J. Math. Phys.* **27**, 1271–1283 (May 1986).

[65] Peetre, J., [*New thoughts on Besov spaces*], Duke University Mathematics Series, No. 1, Mathematics Department, Duke University (1976).

[66] Triebel, H., [*Theory of Function Spaces.*], vol. 78 of *Monographs in Mathematics*, Birkhäuser, Basel (1983).

[67] Triebel, H., [*Theory of Function Spaces II*], Monographs in Mathematics 84, Birkhäuser, Basel (1992).

[68] Triebel, H., [*The Structure of Functions.*], Birkhäuser, Basel (2001).

[69] Triebel, H., [*Theory of Function Spaces III*], vol. 100 of *Monographs in Mathematics*, Birkhäuser (2006).

[70] Peetre, J., "Paracommutators and minimal spaces.," in [*Operators and Function Theory, Proc NATO Adv Study Inst, Lancaster/Engl 1984, NATO ASI Ser, Ser C 153, 163-224*], (1985).

[71] Arazy, J., Fisher, S., and Peetre, J., "Möbius invariant spaces of analytic functions.," in [*Complex Analysis I, Proc Spec Year, College Park/Md 1985-86, Lect Notes Math 1275, 10-22*], (1987).

[72] Arazy, J., Fisher, S. D., and Peetre, J., "Hankel operators on weighted Bergman spaces.," *Amer. J. Math.* **110**(6), 989–1053 (1988).

[73] Coifman, R. R. and Rochberg, R., "Representation theorems for holomorphic and harmonic functions in *lspp*.," *Ast'erisque* **77**, 11–66 (1980).

[74] Heil, C. and Walnut, D. F., "Continuous and discrete wavelet transforms," *SIAM Rev.* **31**, 628–666 (1989).

[75] Walnut, D. F., *Weyl-Heisenberg wavelet expansions: Existence and stability in weighted spaces.*, PhD thesis, University of Maryland, College Park, College Park, MD (1989).

[76] Heil, C., "Wavelets and frames," in [*Signal Processing Part I: Signal Processing Theory, Proc Lect, Minneapolis, MN (USA) 1988, IMA Vol Math Appl 22*], 147–160 (1990).

[77] Walnut, D. F., [*An Introduction to Wavelet Analysis.*], Birkhäuser (2002).

[78] Heil, C., [*A Basis Theory Primer. Expanded ed.*], Applied and Numerical Harmonic Analysis, Basel: Birkhäuser (2011).

[79] Donoho, D. L. and Stark, P. B., "Uncertainty principles and signal recovery," *SIAM J. Appl. Math.* **48**(3), 906–931 (1989).

[80] Feichtinger, H. G., "Coherent frames and irregular sampling," in [*Recent Advances in Fourier Analysis and its Applications, Proc NATO/ASI, IL Ciocco/Italy 1989*], Byrnes, J. and Byrnes, J., eds., *NATO ASI Ser., Ser. C* **315**, 427–440, Kluwer Acad. Publ. (1989).

[81] Gröchenig, K., "Describing functions: atomic decompositions versus frames," *Monatsh. Math.* **112**(3), 1–41 (1991).